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Non-parallel flow corrections for the stability of shear flows

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The proper corrections for non-parallel flow to the eigenvalues for small disturbances on a nearly parallel shear flow have been determined through a perturbation about the parallel flow solutions. The resulting shifts in the neutral stability curves have been calculated for the Blasius boundary layer, for the two-dimensional jet, and for the two-dimensional flat-plate wake.

1. Introduction

Linear stability analyses of shear flows usually treat the basic flow as a quasi-parallel flow. However, most flows are not truly parallel, and the effect of the parallel assumption on the analyses has not been adequately investigated. One approach (e.g. Cheng 1953) has been to treat the basic flow as homogeneous in the streamwise direction, but to include a cross-flow component in the basic flow. This *ad hoc* approach is not very satisfying: commenting on it, Tatsumi & Kakutani (1958) remarked, 'To treat the stability problem of the non-parallel flow in a more satisfactory manner seems to be beyond the scope of the existing theory of hydrodynamical stability'. Lanchon & Eckhaus (1964) examined the effect of non-parallel flow using a formal expansion, and then studied the asymptotic (high Reynolds number) solutions. They found that the quasi-parallel treatment is a proper first approximation for boundary-layer flows, but that, for the more rapidly spreading jet flow, the non-parallel feature must be included in the 'viscous' solutions. They presented no quantitative results. Ko & Lessen (1969) used an *ad hoc* argument to add an extra term to the wave-number to make a non-parallel correction for the jet flow. Barry & Ross (1970) studied more formally the effect of increasing thickness on stability, using a modified Orr–Sommerfeld equation (the argument by which their equation was obtained is incorrect, as will be discussed later).

It is the purpose of the present work to present a proper formal expansion for the linear stability problem for non-parallel flows, and to obtain quantitative results therefrom. The analysis involves a systematic perturbation about the parallel flow solution, and hence provides a rational basis for correcting for weakly non-parallel effects.

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2. Analysis

We treat incompressible fluid flow with constant viscosity, here limiting ourselves to two-dimensional disturbances. For such flows, the motion is described by the vorticity equation (Batchelor 1967)

$$\nabla^2\psi_t+\psi_y\nabla^2\psi_x-\psi_x\nabla^2\psi_y-\frac{1}{R}\nabla^4\psi = 0. \tag{2.1}$$

Here the stream function ψ is defined by

$$u = \psi_y, \quad v = -\psi_x, \tag{2.2}$$

where u and v are the velocity components in the streamwise (x) and cross-stream (y) directions. All quantities are presumed to be normalized on suitable reference length and velocity scales δ_r and u_r, respectively, and R is the Reynolds number based on these scales, $R = u_r\delta_r/\nu$. Let $\overline{\psi}(x,y)$ represent the stream function of the basic flow field, and $\psi'(x,y,t)$ represent the stream function for the disturbance field. Then, setting $\psi = \overline{\psi}+\psi'$, using the fact that $\overline{\psi}$ satisfies (2.1), and neglecting second-order terms in ψ', (2.1) yields the linearized disturbance equation for ψ',

$$\nabla^2\psi_t'+\overline{\psi}_y\nabla^2\psi_x'+\psi_y'\nabla^2\overline{\psi}_x-\overline{\psi}_x\nabla^2\psi_y'-\psi_x'\nabla^2\overline{\psi}_y-\frac{1}{R}\nabla^4\psi' = 0. \tag{2.3}$$

The boundary conditions are of the form

$$\psi_x' = \psi_y' = 0 \quad \text{at solid boundaries,} \tag{2.4a}$$

$$\psi' \to 0 \quad \text{moving outward in a uniform flow.} \tag{2.4b}$$

Since the coefficients are independent of time, we may seek normal-mode solutions ψ' proportional to $\exp\{i\omega t\}$. The eigenvalue ω is of primary interest, with the stability question resting on the sign of ω_i.

Let us assume that $\overline{\psi}$ varies 'slowly' with x, so that a Taylor series about some point x_0 will provide a good representation of $\overline{\psi}$ in the neighbourhood of x_0. Then, we presume that we may write this expansion in the form

$$\overline{\psi} = \overline{\psi}_0(y)+\epsilon\overline{\psi}_1(y)\,\xi+o(\epsilon). \tag{2.5}$$

Here $$\overline{\psi}_0 = \overline{\psi}(x_0,y), \quad \epsilon\overline{\psi}_1(y) = \left.\frac{\partial\psi}{\partial x}\right|_{x_0}, \quad \xi = x-x_0.$$

The small parameter ϵ will depend on the flow under study: e.g. for the Blasius boundary-layer flow, ϵ may be taken as $1/R = 1/(x_0U_\infty/\nu)^{\frac{1}{2}}$ and the terms $o(\epsilon)$ are in fact $O(\epsilon^2\ln\epsilon)$.

In the expansions that follow we shall treat ϵ and R as independent parameters. However, in some cases they will in fact be related. By treating them as independent we are in effect solving the problem in the ϵ, R plane, where in reality we need only to solve the problem on a line. The real solution is therefore contained in the family of fictious extensions over all ϵ, R.

It is particularly important to retain the viscous term in the first approximation to (2.3). Although it appears to be $O(1/R)$ smaller than the inertial terms, it is known to be important near the critical layer (Lin 1955) and near the wall.

Barry & Ross (1970) proposed a modified first approximation based on inclusion of some of the terms involving $\overline{\psi}_x$ in (2.3). For Blasius flow, $\overline{\psi}_x = O(1/R)$; hence they argued that, if viscous terms are retained, so should the $\overline{\psi}_x$ terms. However, the viscous terms are only important near the critical layer, in a region of thickness $O(R^{-\frac{1}{3}})$ (Lin 1955). In this region the viscous terms are actually $O(R^{+\frac{1}{3}})$, since $\partial/\partial y = O(R^{\frac{1}{3}})$; thus $\overline{\psi}_x \nabla^2 \psi'_y$ is $O(1)$, and should in fact be neglected in the first approximation. For such flows, the $\overline{\psi}_x$ may be dropped for the first approximation, which leads to the conventional Orr–Sommerfeld theory. Thus the formal expansions we shall develop, in which the Orr–Sommerfeld equation will give the first approximation, will be proper for all flows in which $\overline{\psi}_x = O(R^{-\frac{2}{3}})$ as $R \to \infty$. These considerations were outlined previously by Lanchon & Eckhaus (1964).

We want to construct a proper expansion of the linearized eigenvalue problem. For parallel flows the coefficients are also independent of x; hence (2.3) admits normal-mode solutions proportional to $\exp\{i\alpha x\}$, where α is the stream-wise wavenumber. Here we expect the same general behaviour, except that the wavenumber α will vary slowly with x. Accordingly, in the spirit of the WKBJ method, we shall look for normal-mode solutions of the form†

$$\psi' = \hat{\psi}(\xi, y) \exp\{i\alpha(\xi)\,\xi\} \exp\{i\omega t\}. \tag{2.6}$$

For weakly non-parallel flows, we expect that both $\hat{\psi}$ and α should be weak functions of ξ. This suggests an expansion of the form

$$\hat{\psi} = \hat{\psi}_0(y) + \epsilon\hat{\psi}_1(y, \xi) + o(\epsilon), \tag{2.7a}$$

$$\alpha = \alpha_0 + \epsilon\alpha_1(\xi) + o(\epsilon), \tag{2.7b}$$

$$\omega = \omega_0 + \epsilon\omega_1 + o(\epsilon). \tag{2.7c}$$

Note that the non-parallel feature is expected to alter the eigenvalue ω, even at the point $\xi = 0$, and $\epsilon\omega_1$ gives this first-order correction to the eigenvalue of the 'local' velocity profile arising from the non-parallel feature of the basic flow.

Now, suppose some fixed point x_0 is chosen, and the non-parallel flow corrections are sought for a disturbance which at point x_0 ($\xi = 0$) is periodic with wavenumber α_0. In this problem no correction to α at $\xi = 0$ would be imposed. Hence, $\alpha_1(0) = 0$. Since the $\overline{\psi}$ expansion to order ϵ involves only the first power of ξ, it is sufficient to take

$$\alpha = \alpha_0 + \epsilon\alpha_1\xi + o(\epsilon). \tag{2.7d}$$

Similarly, to $O(\epsilon)$, $\hat{\psi}_1$ need only be linear in ξ; hence

$$\hat{\psi}_1 = \hat{\psi}_{10}(y) + \xi\hat{\psi}_{11}(y). \tag{2.7e}$$

Higher order analyses would involve terms quadratic in ξ.

† If we use the WKBJ method directly, as suggested by Benney & Rosenblat (1964), and write

$$\psi = \hat{\psi}(x, y) \exp\left\{i \int_{x_0}^{x} \alpha(x')\,dx'\right\} \exp\{i\omega t\},$$

$$\alpha(x) = \alpha_0 + \epsilon\xi\alpha_1^+ + \ldots,$$

then $\alpha_1^+ = 2\alpha_1$ and the analysis is unchanged.

With the expansion as formulated thus far, we can determine the modifications $\epsilon\alpha_1$, $\epsilon\omega_1$ and $\epsilon\hat{\psi}_1$ at fixed α_0 and R. If we wish to determine the resulting shift in the neutral stability curve, it is also convenient to expand R as

$$1/R = b_0 + \epsilon b_1 + o(\epsilon). \tag{2.7f}$$

For Blasius flow, the $o(\epsilon)$ terms in the above equation are all $O(\epsilon^2 \ln \epsilon)$, rather than $O(\epsilon^2)$. Then, b_0 is the reciprocal of the Reynolds number at which disturbances of wavenumber α_0 are neutral in the parallel flow analysis. Knowledge of b_1 will permit calculation of the shift in the neutral Reynolds number produced by the non-parallel aspect of the basic flow. The wave speed c becomes

$$c = -\frac{\omega}{\alpha} = c_0 + \epsilon c_0 \frac{\omega_1}{\omega_0} - \epsilon\xi c_0 \frac{\alpha_1}{\alpha_0} + o(\epsilon) = c_0 + \epsilon(c_1 + \xi c_{11}) + o(\epsilon). \tag{2.7g}$$

When we substitute the expansions (2.7) into (2.3), solvability conditions will yield values for α_1, and a linear relationship between ω_1 and b_1. If we set $b_1 = 0$, we obtain ω_1 (complex); hence the alteration in the eigenvalue for disturbances of wavenumber α_0 at fixed $R = 1/b_0$. Alternatively, suppose we choose α_0 and b_0 such that $\omega_{1i} = 0$ (i.e. we select a point on the neutral stability curve). Then, if we require both b_1 and ω_1 to be real, we can calculate them both, and so deduce the shift in the neutral Reynolds number for disturbances of wavenumber α_0. In general, α_1 will emerge complex. Then $\epsilon\alpha_{1r}\xi$ describes the change in wavenumber as the disturbance moves downstream, and $\exp\{-\epsilon\alpha_{1i}\xi\}$ describes the downstream change in the disturbance amplitude. The distortion of the eigenfunction at x_0 due to non-parallel effects is described by $\epsilon\hat{\psi}_{10}(y)$, and the additional distortion as the disturbance moves downstream is indicated by $\epsilon\hat{\psi}_{11}(y)\,\xi$.

With this preview of the direction of the analysis, we proceed with the details. Substituting (2.5)–(2.7) into (2.3), and collecting terms of like orders of ϵ and ξ, one obtains a sequentially solvable set of ordinary differential equations. For $O(\epsilon^0\xi^0)$,

$$L(\hat{\psi}_0) = 0, \tag{2.8}$$

where
$$L = i\{[\omega_0 + \alpha_0 \overline{\psi}_0'](D^2 - \alpha_0^2) - \alpha_0 \overline{\psi}_0'''\} - b_0(D^2 - \alpha_0^2)^2,$$
$$D = d/dy.$$

This is the familiar Orr–Sommerfeld equation of parallel shear flow stability theory (Lin 1955). With homogeneous boundary conditions, (2.8) defines an eigenvalue problem for the eigenvalue ω_0. For $O(\epsilon^1\xi^1)$,

$$L(\hat{\psi}_{11}) = -\alpha_1 G + H, \tag{2.9}$$

where
$$G = \{2i[\overline{\psi}_0'(D^2 - 3\alpha_0^2) - \overline{\psi}_0''' - 2\omega_0\alpha_0] + 8b_0\alpha_0(D^2 - \alpha_0^2)\}\,\hat{\psi}_0,$$
$$H = -i\alpha_0[\overline{\psi}_1'(D^2 - \alpha_0^2) - \overline{\psi}_1''']\,\hat{\psi}_0.$$

For $O(\epsilon^1\xi^0)$,
$$L(\hat{\psi}_{10}) = -\omega_1 P - b_1 Q + S, \tag{2.10}$$

where
$$P = i(D^2 - \alpha_0^2)\,\hat{\psi}_0, \quad Q = -(D^2 - \alpha_0^2)^2\,\hat{\psi}_0,$$
$$S = 2\omega_0(\alpha_0\hat{\psi}_{11} + \alpha_1\hat{\psi}_0) - \overline{\psi}_0'(D^2 - 3\alpha_0^2)\,\hat{\psi}_{11} + 6\alpha_0\alpha_1\overline{\psi}_0'\hat{\psi}_0 + \overline{\psi}_1 D(D^2 - \alpha_0^2)\,\hat{\psi}_0$$
$$+ \overline{\psi}_0'''\,\hat{\psi}_{11} - \overline{\psi}_1'' D\hat{\psi}_0 + 4ib_0[\alpha_1(D^2 - 3\alpha_0^2)\,\hat{\psi}_0 + \alpha_0(D^2 - \alpha_0^2)\,\hat{\psi}_{11}].$$

The boundary conditions (2.4) yield

$$\hat{\psi}_0 \to 0, \quad \hat{\psi}_{10} \to 0, \quad \hat{\psi}_{11} \to 0, \quad \text{moving outward in a uniform stream;} \tag{2.11a}$$

$$\hat{\psi}_0 = D\hat{\psi}_0 = 0, \quad \hat{\psi}_{10} = D\hat{\psi}_{10} = 0, \quad \hat{\psi}_{11} = D\hat{\psi}_{11} = 0, \quad \text{at solid boundaries.} \tag{2.11b}$$

In problems where the eigensolutions are either symmetric or antisymmetric, (2.11b) is replaced by symmetry or antisymmetry conditions.

To carry out the calculations, we must first solve the conventional eigenvalue problem posed by (2.8) and (2.11). This gives a point α_0, b_0, ω_0, and the corresponding eigenfunction $\hat{\psi}_0(y)$ about which the expansion is made. The associated adjoint eigenfunction $\Phi(y)$ will be useful. It satisfies the adjoint equation (Stuart 1960)

$$\mathscr{L}(\Phi) = 0, \tag{2.12}$$

where $$\mathscr{L} = i\{[\omega_0 + \alpha_0 \overline{\psi}_0'](D^2 - \alpha_0^2) + 2\alpha_0 \overline{\psi}_0'' D\} - b_0(D^2 - \alpha_0^2)^2,$$

and boundary conditions identical to (2.11). Moreover, the adjoint problem has the same eigenvalue. The boundary conditions on Φ are

$$\Phi \to 0 \quad \text{moving outward in a uniform stream;} \tag{2.13a}$$

$$\Phi = D\Phi = 0 \quad \text{on a solid boundary.} \tag{2.13b}$$

In problems where the eigensolutions are symmetric or antisymmetric about $y = 0$, (2.13b) is replaced by symmetry or antisymmetry conditions. Details of this computation are described in Ling & Reynolds (1971).

2.1. *Solvability condition*

The adjoint eigenfunction has several important properties. It is defined so that, if f and g are any two functions satisfying (2.11),

$$\int_1^2 fL(g)\,dy = \int_1^2 g\mathscr{L}(f)\,dy, \tag{2.14}$$

where 1 and 2 denote the boundaries of the flow. Suppose one is interested in solving an inhomogeneous equation of the form (e.g. (2.9) and (2.10))

$$L(h) = M, \tag{2.15a}$$

with boundary conditions (2.11) on h, and L such that $L(\hat{\psi}_0) = 0$ (i.e. an eigensolution exists). It follows from (2.14) that (2.15a) cannot be solved unless

$$\int_1^2 M\Phi\,dy = 0. \tag{2.15b}$$

This is the solvability condition which will be invoked in the determination of α_1, ω_1 and b_1. However, if (2.15b) holds, the solution to (2.15a) is not unique, for we may add to any solution h an arbitrary multiple of $\hat{\psi}_0$ to produce a new solution of (2.15a). This will be discussed in § 2.3.

Since h can be expressed as the sum of a particular solution, a well-behaved homogeneous solution and a growing homogeneous solution, we have

$$h = h_p + a_1\hat{\psi}_{0g} + a_2\hat{\psi}_0, \tag{2.16}$$

where h_p is the particular solution, $\hat{\psi}_0$ the Orr–Sommerfeld eigensolution, and $\hat{\psi}_{0g}$ a second solution to the homogeneous equation. a_1 can be found by applying the boundary condition, and a_2 will be obtained by using an orthogonality condition to be described in § 2.2.

Suppose we have solved numerically the eigenvalue problem (2.8), determined ω_0 for values of α_0 and b_0, and tabulated the eigenfunction and its associated adjoint eigenfunction. Turning to the $\hat{\psi}_{11}$ problem (2.9), we see that the inhomogeneous terms on the right contain only known functions and the unknown parameter α_1. A solution satisfying the boundary conditions is impossible unless the solvability condition (2.15b) is satisfied. Hence, we must take

$$\alpha_1 = \int_1^2 H\Phi\, dy \bigg/ \int_1^2 G\Phi\, dy. \tag{2.17}$$

We can then compute numerically a solution to (2.9) satisfying the boundary conditions, and add to that solution a multiple of $\hat{\psi}_0$, to satisfy the orthogonality condition (to be discussed).

With α_1 and $\hat{\psi}_{11}$ in hand, we turn to (2.10), and note that the inhomogeneous terms contain only known functions and the unknown constants ω_1 and b_1. The solvability condition requires that

$$\omega_1 \int_1^2 P\Phi\, dy + b_1 \int_1^2 Q\Phi\, dy = \int_1^2 S\Phi\, dy, \tag{2.18}$$

or

$$C_1\omega_1 + C_2 b_1 = C_3. \tag{2.19a}$$

Now, if we wish the eigenvalue perturbation at fixed Reynolds number, we simply set $b_1 = 0$ and determine ω_1 from (2.18). Alternatively, if we wish to determine the Reynolds number change required to hold ω_i fixed, we note that $\omega_{1i} = b_{1i} = 0$; hence

$$C_1^*\omega_1 + C_2^* b_1 = C_3^*. \tag{2.19b}$$

Equations (2.19) may be solved for ω_1 and b_1. With these constants, we may proceed to solve numerically for $\hat{\psi}_{10}$, again adding a multiple of $\hat{\psi}_0$ by using the orthogonality condition, and the problem is then completely solved to $O(\epsilon)$. For details and examples of such calculations, see Ling & Reynolds (1971).

2.2. *Orthogonality condition*

The solution of the inhomogeneous equation (2.15a) can be viewed as an expansion in terms of the eigenfunctions of $L(\hat{\psi}_0) = 0$, taking the complete set of eigenvalues $\omega_0^{[n]}$,

$$h = \sum_{n=0}^{\infty} C^{[n]}\hat{\psi}_0^{[n]} = C^{[0]}\hat{\psi}_0^{[0]} + \sum_{n=1}^{\infty} C^{[n]}\hat{\psi}_0^{[n]},$$

where $\hat{\psi}_0^{[0]}$ refers to the first eigenfunction. Based on arguments for the uniqueness of the solution (to be discussed), $C^{[0]}$ must be zero. Hence,

$$h = \sum_{n=1}^{\infty} C^{[n]}\hat{\psi}_0^{[n]}. \tag{2.20}$$

The eigenfunctions $\hat{\psi}_0^{[n]}$ satisfy (2.8) with ω_0 replaced by $\omega^{[n]}$, which may be written as

$$L(\hat{\psi}^{[n]}) = L_1(\hat{\psi}_0^{[n]}) + \omega^{[n]} L_2(\hat{\psi}_0^{[n]}) = 0, \qquad (2.12a)$$

$$L_2 = i(D^2 - \alpha_0^2).$$

Also,
$$\mathscr{L}(\Phi) = \mathscr{L}_1(\Phi) + \omega_0 L_2(\Phi) = 0, \qquad (2.21b)$$

where $\mathscr{L}_1$ is adjoint to L_1. Multiplying (2.21*a*) by Φ, and (2.21*b*) by $\hat{\psi}^{[n]}$, integrating and subtracting, one finds

$$\int_1^2 \Phi(D^2 - \alpha_0^2)\, \hat{\psi}^{[n]}\, dy = 0 \quad \text{if} \quad \omega^{[n]} \neq \omega_0.$$

Alternatively, integrating by parts, using the boundary conditions

$$\int_1^2 \hat{\psi}^{[n]}(D^2 - \alpha_0^2)\, \Phi\, dy = 0 \quad (n > 0).$$

Hence, using (2.20),

$$\int_1^2 h(D^2 - \alpha_0^2)\, \Phi\, dy = 0, \quad \int_1^2 \Phi(D^2 - \alpha_0^2)\, h\, dy = 0. \qquad (2.22)$$

This condition will be used to suppress $\hat{\psi}_0$ from $\hat{\psi}_{11}$ and $\hat{\psi}_{10}$, as the discussion following suggests is required. Details of this computation are described in Ling & Reynolds (1971).

2.3. *Uniqueness of solution*

The rationale behind the suppression of $\hat{\psi}_0$ from the higher order functions deserves some comment. Suppose we add to $\hat{\psi}_{11}$ a multiple A of $\hat{\psi}_0$; then we could write the ψ' expansion as

$$\psi' = [\hat{\psi}_0(1 + A\epsilon\xi) + \epsilon(\hat{\psi}_{10} + \xi\hat{\psi}_{11})] \exp\{i(\alpha_0\xi + \epsilon\alpha_1\xi^2) + i\omega t\} + \dots. \qquad (2.23)$$

To $O(\epsilon)$ we could rewrite the first term, and obtain

$$\psi' = [\hat{\psi}_0 + \epsilon(\hat{\psi}_{10} + \xi\hat{\psi}_{11}) + o(\epsilon)] \exp\{i[(\alpha_0 - i\epsilon A)\,\xi + \epsilon\alpha_1\xi^2 + \dots] + i\omega t\}. \quad (2.24)$$

In this form we see that the term containing A has the same effect as a change in the *wavenumber* at $\xi = 0$. But it was our intent to examine the effect of non-parallelism on disturbances which at x_0 have a given wavenumber α_0. Hence, we should prevent any additional perturbations in wavenumber from creeping in at $\xi = 0$, which requires that we choose $A = 0$.

The function $\hat{\psi}_{10}$ might also contain an arbitrary multiple of $\hat{\psi}_0$. Suppose we add $B\hat{\psi}_0$ to $\hat{\psi}_{10}$, then the expansion could be written as

$$\psi' = [\hat{\psi}_0(1 + \epsilon B) + \epsilon(\hat{\psi}_{10} + \xi\hat{\psi}_{11}) + \dots] \exp\{i(\alpha\xi + \omega t)\}, \qquad (2.25)$$

and the term containing B has the same effect as a change in amplitude of the basic eigenfunction $\hat{\psi}_0$. Since in this linear problem the amplitude is arbitrary, we may set $B = 0$ without loss of generality.

The choice of ϵ and $\overline{\psi}_1$ in (2.5) may seem arbitrary. Study of (2.9) and (2.10) shows that the products $\epsilon\alpha_1$ and $\epsilon\omega_1$ are independent of the portion of the constant multiplying ξ in (2.5) that is assigned to ϵ; hence the results of the analysis are independent of this arbitrary choice. One should note that the first approximation is indeed provided by (2.8), the Orr–Sommerfeld equation as normally used in quasi-parallel analysis (Lanchon & Eckhaus 1964).

3. Numerical procedure

The numerical procedure for integration of (2.8)–(2.10) and (2.12) is patterned on that described by Reynolds & Potter (1967) and Reynolds (1969). The solution is carried out numerically using a fourth-order linear algorithm with Kaplan filtering (Lee & Reynolds 1967). Starting at the outer edge of the shear layer, two homogeneous solutions are first constructed numerically for the adjoint problem (2.12) with specified α_0 and trial values of c_0 and R. These both satisfy the boundary conditions far from the shear layer (2.13*a*). Kaplan's filtering technique produces two linearly independent solutions, to which we refer as the 'well-behaved' and the 'growing' solution. The boundary conditions at the end of the integration range are, alternatively (see 2.13*b*),

$$\Phi = D\Phi = 0 \quad \text{for a solid wall,} \tag{3.1a}$$

or

$$D\Phi = D^3\Phi = 0 \quad \text{on an axis of symmetry in } \Phi, \tag{3.1b}$$

or

$$\Phi = D^2\Phi = 0 \quad \text{on an axis of antisymmetry in } \Phi. \tag{3.1c}$$

To satisfy the two conditions, a linear combination of the two solutions is formed that satisfies the second of the two conditions, and the first is satisfied automatically for eigensolutions. An iteration scheme is used to vary both c_0 and R_0 until the first boundary condition is satisfied, and then Φ is the desired eigensolution. With the eigenvalue in hand, we next solve (2.8) in the same manner; the growing solution of (2.8) is stored for subsequent use in the solution of the inhomogeneous equations.

After α_1 has been obtained from (2.17) using numerical integration, (2.9) is integrated by using a proper starting solution, and the same procedure used for solving (2.8) and (2.12). We compute a well-behaved particular solution, and form the final solution by adding an appropriate multiple of the growing homogeneous solution previously computed. To determine this multiple, the second boundary condition of (3.1) is again employed, and the first is automatically satisfied if (2.17) is satisfied. Then, (2.22) is used to suppress $\hat{\psi}_0$. Equation (2.18) is then used to determine ω_1. These procedures introduce the proper amount of both into the final solution. Then (2.10) is solved by the same procedure.

4. Results and discussion

4.1. *Blasius flow*

For the Blasius boundary layer (Schlichting 1968, p. 125), the dimensional stream function is

$$\overline{\psi}^* = (\nu x U_\infty)^{\frac{1}{2}} f(\eta), \tag{4.1}$$

where

$$\eta = y(U_\infty/\nu x)^{\frac{1}{2}}, \tag{4.2}$$

and $f(\eta)$ is given by the solution of

$$2f''' + f'f'' = 0, \qquad f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0{\cdot}33206. \tag{4.3}$$

f and its derivatives, and therefore the stream functions and their derivatives, can be found easily by solving (4.3) numerically.

Using the leading-edge expansion of Van Dyke (1964), and expanding about x_0, one has

$$\overline{\psi}^* = (\nu x_0 U_\infty)^{\frac{1}{2}} \left[f(\eta_0) + \frac{1}{2}\left(\frac{U_\infty \nu}{x_0}\right)^{\frac{1}{2}} (f(\eta_0) - \eta_0 f'(\eta_0))\,(x - x_0) + \frac{\nu}{x_0 U_\infty} \ln\left(\frac{\nu}{x_0 U_\infty}\right)^{\frac{1}{2}} (1/\sqrt{2} f_{32}(\eta_0)) + \dots \right], \tag{4.4}$$

where
$$\eta_0 = Y = y\left(\frac{U_\infty}{\nu x_0}\right)^{\frac{1}{2}} \tag{4.5}$$

and f_{32} is a function defined by Van Dyke.

Normalizing with a characteristic length $\delta_r = (\nu x_0/U_\infty)^{\frac{1}{2}}$ and a characteristic velocity $u_r = U_\infty$, and introducing

$$R = u_r \delta_r/\nu = (x_0 U_\infty/\nu)^{\frac{1}{2}}, \tag{4.6}$$

the dimensionless stream function becomes (compare (2.5))

$$\overline{\psi} = f(Y) + \frac{1}{R}\{\tfrac{1}{2}[f(Y) - Yf'(Y)]\}\,\xi + o(1/R). \tag{4.7}$$

Hence, in (2.5) we may take

$$\overline{\psi}_0(Y) = f(Y), \tag{4.8}$$

$$\overline{\psi}_1(Y) = \tfrac{1}{2}[f(Y) - Yf'(Y)], \tag{4.9}$$

$$\epsilon = 1/R. \tag{4.10}$$

The neutral Reynolds number for a given α_0 is (see (2.7*f*))

$$R_N = 1/(b_0 + \epsilon b_1 + o(\epsilon)) = \frac{1}{b_0}\left[1 - \epsilon\frac{b_1}{b_0} + o(\epsilon)\right],$$

with
$$\left.\begin{aligned} &R_0 = \frac{1}{b_0}, \quad R_1 = -\frac{b_1}{b_0^2}, \\ &R_N = R_0 + \epsilon R_1 + o(\epsilon) = R_0 + \frac{1}{R_0} R_1 + o(\epsilon). \end{aligned}\right\} \tag{4.11}$$

The sequence of problems (2.12), (2.8)–(2.10) was solved using $\overline{\psi}_0$ and $\overline{\psi}_1$, as given above. Table 1 gives the computational results. Figures 2–5 show the functions Φ, $\hat{\psi}_0$, $\hat{\psi}_{11}$, $\hat{\psi}_{10}$ for $\alpha_0 = 0{\cdot}172$ and $R_0 = 302{\cdot}4$; and figure 6 shows the streamwise growth of disturbance for the same α_0 and R_0. The calculations show that $R_1 < 0$ at the critical R_0, indicating a *reduction* in the critical Reynolds number due to the non-parallel flow effect. However, the change in critical Reynolds number is very small (about 0·1 %), and we conclude that the parallel flow model does a remarkably good job (figure 1). Since $\alpha_{1r} < 0$, the wavenumber will shrink slightly in the downstream direction (i.e. the wavelength will grow). At the critical point $\alpha_{1i} < 0$, which means that a disturbance which is marginally stable in time will show a slight downstream amplification (figure 6).

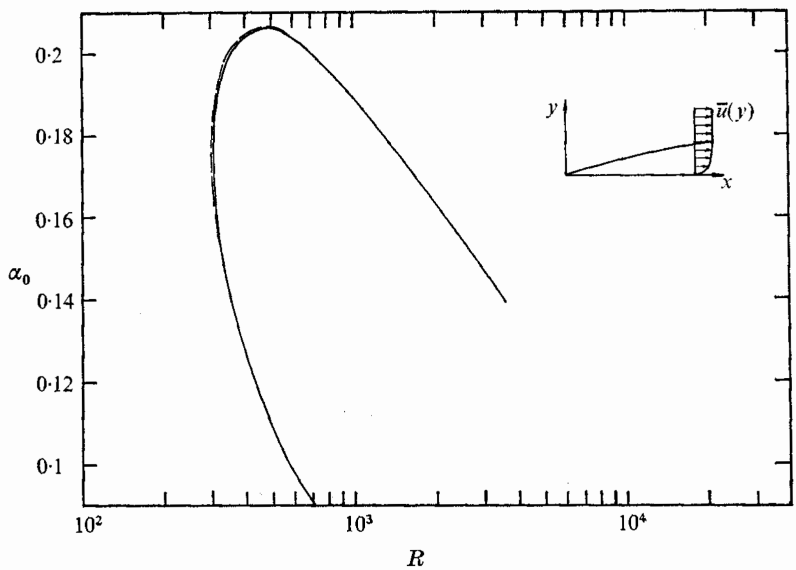

FIGURE 1. Neutral stability curves for Blasius flow: ——, parallel flow; – – – –, non-parallel flow.

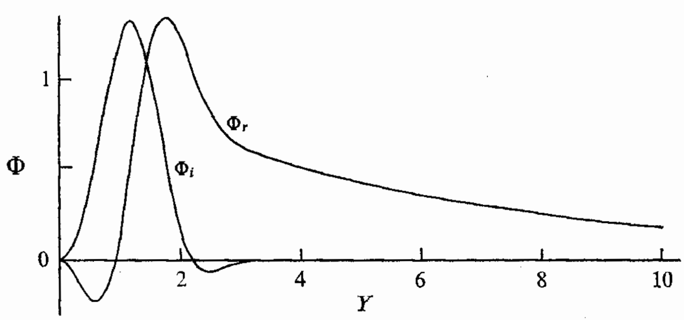

FIGURE 2. Adjoint eigenfunction Φ for Blasius flow at $\alpha = 0{\cdot}172$, $R_0 = 302{\cdot}4$.

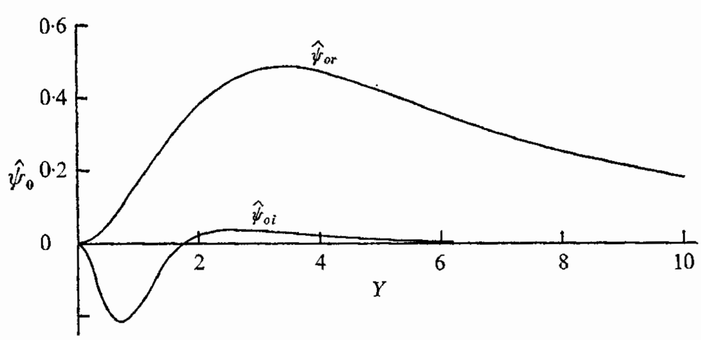

FIGURE 3. Eigenfunction $\hat{\psi}_0$ for Blasius flow at $\alpha = 0{\cdot}172$, $R_0 = 302{\cdot}4$.

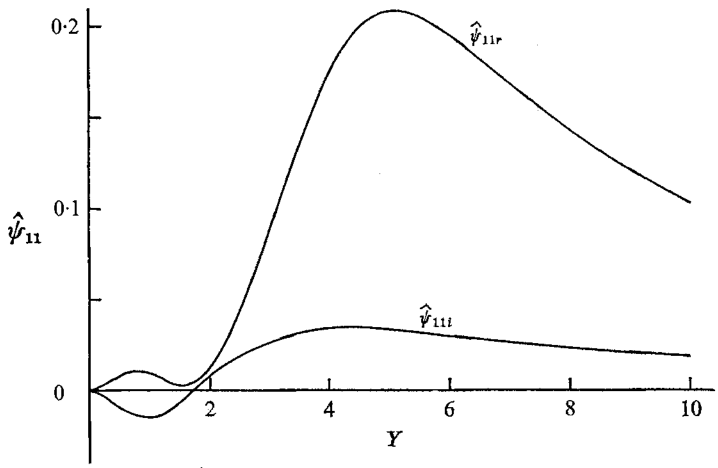

FIGURE 4. $\hat{\psi}_{11}$ for Blasius flow at $\alpha_0 = 0{\cdot}172$, $R_0 = 302{\cdot}4$.

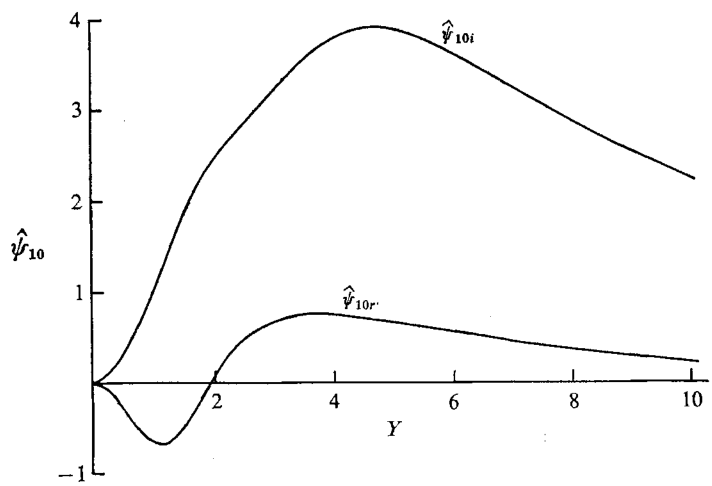

FIGURE 5. $\hat{\psi}_{10}$ for Blasius flow at $\alpha_0 = 0{\cdot}172$, $R_0 = 302{\cdot}4$.

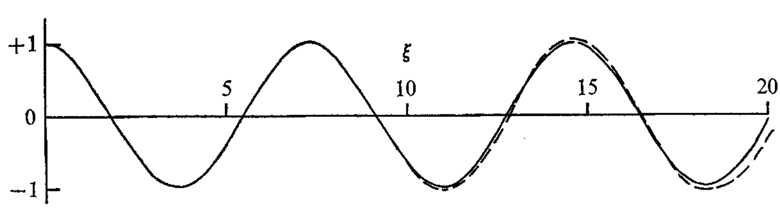

FIGURE 6. Streamwise growth of disturbance for Blasius flow at $\alpha_0 = 0{\cdot}072$, $R_0 = 302{\cdot}4$: ——, $\mathrm{Re}(\exp\{i\alpha_0\xi\})$; – – – –, $\mathrm{Re}(\exp\{i\alpha\xi\})$.

Parallel flow eigensolution				Non-parallel flow corrections							
				Perturbation in neutral curve						Perturbation at fixed R	
α_0	R_0	$-\omega_0 \times 10^2$	c_0	$\alpha_1 \times 10^3$	ω_1	b_1	R_1	c_{10}	$c_{11} \times 10^2$	ω_1	$R_0 + \frac{1}{R_0} R_1$
0·09	699·1	2·76	0·307	$-2·55-5·05i$	$-0·228$	0·917	-4485	2·53	$0·870+1·72i$	$-0·204-1·79i$	692·7
0·116	448·8	4·00	0·345	$-2·73-5·28i$	$-0·238$	0·0133	-2694	2·05	$0·814+1·57i$	$-0·206-0·0224i$	442·8
0·144	340·0	5·40	0·375	$-2·57-4·66i$	$-0·242$	0·0168	-1950	0·68	$0·669+1·21i$	$-0·201-0·0250i$	334·2
0·172	302·4	6·78	0·394	$-2·05-2·69i$	$-0·242$	0·0196	-1798	1·40	$0·472+0·617i$	$-0·189-0·0251i$	296·5
0·190	310·9	7·60	0·400	$-1·59-0·137i$	$-0·245$	0·0223	-2161	1·29	$0·336+0·0288i$	$-0·175-0·0236i$	303·9
0·200	339·6	7·98	0·399	$-1·56+2·32i$	$-0·260$	0·0272	-3138	1·30	$0·271-0·464i$	$-0·161-0·0216i$	330·4
0·204	591·6	7·56	0·371	$-2·27+10·1i$	$-0·391$	$-0·0108$	3770	0·192	$0·412-1·85i$	$-0·106-0·0107i$	600·0
0·190	956·9	6·53	0·344	$-4·25+13·4i$	$-0·0765$	0·00158	-1444	0·402	$0·768-2·42i$	$-0·0630+0·00534i$	955·4
0·166	1807·8	$-5·12$	0·309	$-7·42+14·7i$	$-0·0675$	0·00548	-17900	0·407	$1·38-2·74i$	$-0·00453+0·0477i$	1800·0

TABLE 1. Computational results for the Blasius flow

4.2. *Two-dimensional laminar jet*

For the two-dimensional laminar jet (Schlichting 1968, p. 170), the dimensional stream function is

$$\bar{\psi}^* = 2\gamma\nu^{\frac{1}{2}}x^{\frac{1}{3}}F(\eta), \tag{4.12}$$

where γ is a constant related to the jet momentum, and

$$\eta = \frac{\gamma y}{3\nu^{\frac{1}{2}}x^{\frac{2}{3}}}, \quad F(\eta) = \tanh\eta.$$

Again we should note that (4.12) is based on the boundary-layer equations, which break down at low Reynolds number.

Expanding about x_0, one gets

$$\bar{\psi}^* = 2\gamma\nu^{\frac{1}{2}}x_0^{\frac{1}{3}}F(\eta_0) + \frac{2\gamma}{3}\nu^{\frac{1}{2}}x_0^{-\frac{2}{3}}[F(\eta_0) - 2\eta_0F'(\eta_0)](x-x_0) + \ldots, \tag{4.13}$$

where

$$\eta_0 = Y = \frac{\gamma y}{3\nu^{\frac{1}{2}}x_0^{\frac{2}{3}}}.$$

Normalizing with a characteristic length $\delta_r = 3\nu^{\frac{1}{2}}x_0^{\frac{2}{3}}/\gamma$, and a characteristic velocity $u_r = u_{\max} = 2\gamma^2/(3x_0^{\frac{1}{3}})$, the dimensionless stream function becomes

$$\bar{\psi} = F(Y) + (2/R)[F(Y) - 2YF']\xi + O(1/R^2), \tag{4.14}$$

where

$$R = u_r\delta_r/\nu.$$

Hence, in (2.5) we may take

$$\bar{\psi}_0(Y) = F(Y) = \tanh Y, \tag{4.15}$$

$$\bar{\psi}_1(Y) = 2F(Y) - 4YF'(Y), \tag{4.16}$$

$$\epsilon = 1/R. \tag{4.17}$$

Solutions were obtained for disturbances symmetric about the centre-line by methods in §3. Tables 2 and 3 give the computational results. Figure 7 shows the non-parallel effect on the neutral stability curve. Figure 12 shows the streamwise growth of disturbance for the same α_0 and R_0.

Note that, at the critical point on the parallel flow neutral curve, R_1 is negative, indicating again the destabilizing effect of the non-parallel flow. However, because R_0 is low here (compared with the Blasius case), ϵ is not particularly small, and the analysis to $O(\epsilon)$ is not sufficient to determine the shift in the critical Reynolds number (see figure 7). For such low Reynolds number flows, the use of the basic flow as given by the boundary-layer equations seems dubious at best. Nevertheless, it is quite clear that the non-parallel flow reduces the critical Reynolds number. The curves of figure 8 were drawn for an α for which the $O(\epsilon)$ analysis is probably adequate. At this point $\alpha_{1r} < 0$, again indicating a lengthening of the wavelength of a neutral disturbance in the flow direction. At this point $\alpha_{1i} > 0$, suggesting a streamwise *reduction* in the amplitude of a marginally stable disturbance. Figures 9 and 10 show the wave speeds and growth rates for the parallel flow and non-parallel flow models at $R \approx 29$ and $R \approx 84$. Note that the correction is quite small at these Reynolds numbers, and smaller at the higher Reynolds numbers. The non-parallel flow effect is most pronounced at low wavenumber, as expected.

Parallel flow eigensolution				Non-parallel flow corrections							
					Perturbation in neutral curve					Perturbation at fixed R	
α_0	R_0	ω_0	c_0	α_1	ω_1	b_1	R_1	c_{10}	c_{11}	ω_1	$R_0+\frac{1}{R_0}R_1$
0·10	4·25	−0·00166	0·0166	$-0·0583-0·114i$	−0·165	6·70	−121	1·65	$0·0101+0·0187i$	$-0·105-0·111i$	—
0·15	4·03	−0·00511	0·0341	$-0·0967-0·127i$	−0·172	3·12	−50·8	1·15	$0·0220+0·0288i$	$-0·128-0·134i$	—
0·20	4·03	−0·0109	0·0546	$-0·135-0·128i$	−0·176	2·12	−34·6	0·880	$0·0369+0·0349i$	$-0·142-0·170i$	—
0·25	4·15	−0·0192	0·0770	$-0·171-0·119i$	−0·181	1·67	−28·9	0·724	$0·0527+0·0368i$	$-0·157-0·214i$	—
0·30	4·34	−0·0301	0·100	$-0·204-0·105i$	−0·186	1·40	−26·3	0·619	$0·0683+0·0352i$	$-0·174-0·259i$	—
0·40	4·87	−0·0592	0·148	$-0·262-0·0615i$	−0·198	1·03	−24·4	0·495	$0·0970+0·0227i$	$-0·213-0·333i$	—
0·50	5·58	−0·0974	0·195	$-0·310-0·00251i$	−0·212	0·771	−24·0	0·423	$0·121+0·00098i$	$-0·253-0·381i$	1·28
0·60	6·47	−0·144	0·239	$-0·350+0·0671i$	−0·224	0·578	−24·3	0·373	$0·140-0·0268i$	$-0·284-0·402i$	2·74
0·75	8·21	−0·226	0·302	$-0·396+0·184i$	−0·234	0·372	−25·1	0·311	$0·159-0·0739i$	$-0·310-0·392i$	5·05
1·00	12·58	−0·393	0·393	$-0·437+0·387i$	−0·230	0·164	−25·9	0·230	$0·172-0·152i$	$-0·303-0·295i$	10·5
1·40	28·25	−0·722	0·515	$-0·419+0·670i$	−0·229	−0·00394	3·14	0·164	$0·154-0·247i$	$-0·225+0·0132i$	28·3
1·75	83·88	−1·06	0·607	$-0·322+0·817i$	−0·368	−0·0800	563·5	0·210	$0·112-0·283i$	$-0·200+0·378i$	90·6

TABLE 2. Computational results for the two-dimensional laminar jet

		Parallel flow eigensolution		Non-parallel flow corrections	Perturbation at fixed R			
R_0	α_0	ω_0	c_0	α_1	ω_1	c_{10}	c_{11}	$c_0+\frac{1}{R_0}c_{10}$
28·2	0·1	0·00573 − 0·0104i	−0·0573 + 0·104i	−0·140 + 0·0821i	−0·475 + 0·642i	4·75 − 6·42i	0·00526 + 0·194i	0·119 − 0·133i
28·2	0·2	−0·0101 − 0·0368i	0·0505 + 0·184i	−0·156 + 0·135i	−0·345 + 0·437i	1·72 − 2·19i	0·163 + 0·110i	0·114 + 0·103i
28·2	0·3	−0·0397 − 0·0610i	0·132 + 0·203i	−0·171 + 0·194i	−0·282 + 0·322i	0·941 − 1·07i	0·207 + 0·0303i	0·167 + 0·163i
28·2	0·5	−0·124 − 0·0947i	0·249 + 0·189i	−0·200 + 0·329i	−0·270 + 0·127i	0·540 − 0·255i	0·224 − 0·0877i	0·269 + 0·180i
28·2	0·75	−0·261 − 0·106i	0·348 + 0·141i	−0·256 + 0·490i	−0·304 − 0·0322i	0·406 + 0·430i	0·211 − 0·179i	0·363 + 0·139i
28·2	1·00	−0·422 − 0·0857i	0·422 + 0·0857i	−0·325 + 0·606i	−0·306 − 0·0970i	0·306 + 0·0970i	0·189 − 0·228i	0·433 + 0·0893i
28·2	1·20	−0·565 − 0·0499i	0·471 + 0·0416i	−0·376 + 0·657i	−0·272 − 0·0764i	0·227 + 0·0637i	0·170 − 0·245i	0·479 + 0·0392i
28·2	1·40	−0·722 + 0i	0·515 + 0i	−0·419 + 0·670i	−0·225 + 0·0132i	0·161 − 0·00946i	0·154 − 0·247i	0·521 + 0·0003i
83·9	0·1	0·00163 − 0·0192i	−0·0163 + 0·192i	−0·0664 + 0·101i	−0·455 + 0·796i	4·55 − 7·96i	0·184 + 0·144i	0·0379 + 0·097i
83·9	0·2	−0·0201 − 0·0484i	0·100 + 0·242i	−0·0661 + 0·152i	−0·309 + 0·687i	1·55 − 3·43i	0·217 + 0·00361i	0·118 + 0·201i
83·9	0·3	−0·0537 − 0·0757i	0·179 + 0·252i	−0·0632 + 0·218i	−0·243 + 0·566i	0·810 − 1·89i	0·221 − 0·0770i	0·188 + 0·232i
83·9	0·75	−0·286 − 0·141i	0·382 + 0·188i	−0·108 + 0·577i	−0·278 + 0·148i	0·371 − 0·198i	0·199 − 0·267i	0·386 + 0·186i
83·9	1·20	−0·598 − 0·1141i	0·498 + 0·0953i	−0·223 + 0·809i	−0·291 + 0·0549i	0·243 − 0·0457i	0·157 − 0·318i	0·501 + 0·0949i
83·9	1·75	−1·06 + 0i	0·607 + 0i	−0·322 + 0·817i	−0·200 + 0·378i	0·114 − 0·216i	0·112 − 0·283i	0·608 − 0·00257i

TABLE 3. Computational results for the two-dimensional laminar jet. Perturbation at fixed R

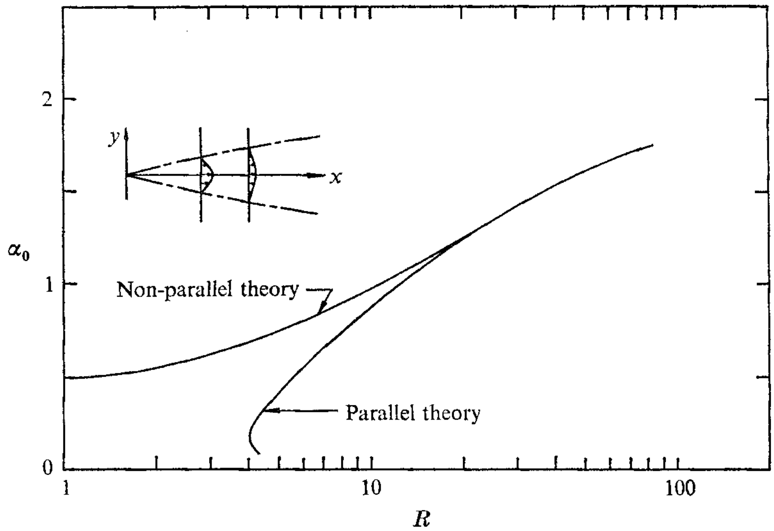

FIGURE 7. Neutral stability curves for the two-dimensional laminar jet.

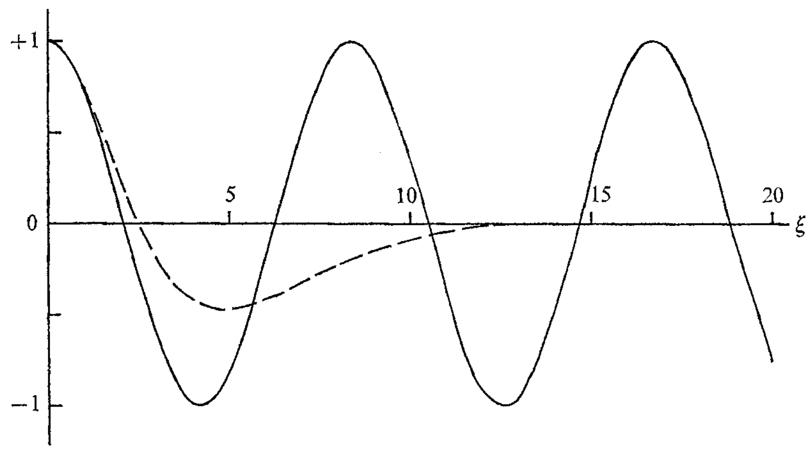

FIGURE 8. Streamwise disturbance behaviour for the two-dimensional laminar jet at $\alpha_0 = 0{\cdot}75$, $R_0 = 8{\cdot}21$: ——, Re (exp $\{i\alpha_0\xi\}$); – – – –, Re (exp $\{i\alpha\xi\}$).

Ko & Lessen (1969) made an *ad hoc* correction for the non-parallel effect on laminar jet stability; their results do not agree with our formal expansion analysis. In essence, they found $a_{1i} > 0$, which is not always the case (tables 2 and 3); in addition, the trends in a_{1i} with Reynolds number suggested by Ko & Lessen are not supported by the present theory.

4.3. *Two-dimensional flat-plate wake*

For the two-dimensional flat-plate wake (Schlichting 1968, p. 166), the dimensional velocity in the x direction is

$$\overline{u}^* = U_\infty\left[1-\frac{0{\cdot}664}{\sqrt{\pi}}\left(\frac{x}{l}\right)^{-\frac{1}{2}}g(\eta)\right], \tag{4.18}$$

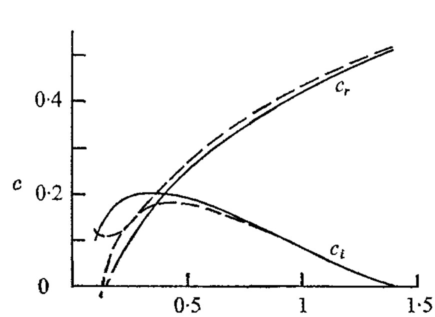

FIGURE 9

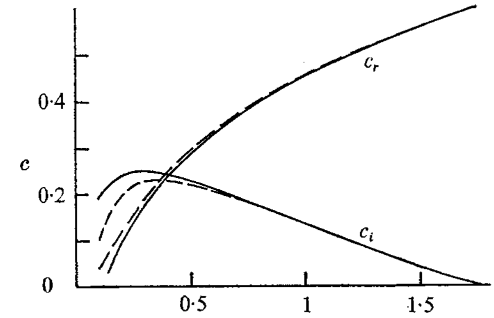

FIGURE 10

FIGURE 9. Wave speed and its correction for the two-dimensional laminar jet at $R_0 = 28{\cdot}2$: ——, parallel flow; ----, non-parallel flow.

FIGURE 10. Wave speed and its correction for the two-dimensional laminar jet at $R_0 = 83{\cdot}9$: ——, parallel flow; ----, non-parallel flow.

where l is the length of the plate, and

$$\eta = y\left(\frac{U_\infty}{\nu x}\right)^{\frac{1}{2}}, \quad g(\eta) = \exp\{-\eta^2/4\}.$$

So the dimensional stream function is

$$\overline{\psi}^* = U_\infty\left[y - \frac{0{\cdot}664}{\sqrt{\pi}}\left(\frac{x}{l}\right)^{-\frac{1}{2}}\left(\frac{\nu x}{U_\infty}\right)^{\frac{1}{2}} F(\eta)\right], \tag{4.19}$$

where
$$F(\eta) = \int_0^\eta g(z)\,dz.$$

This is of course correct only asymptotically, far downstream from the plate, as $(xU_\infty/\nu)^{\frac{1}{2}} \to \infty$:

$$\overline{\psi}^* = (\nu U_\infty x_0)^{\frac{1}{2}}\eta_0 - \frac{0{\cdot}664}{\sqrt{\pi}}(\nu U_\infty l)^{\frac{1}{2}} F(\eta_0) + \frac{1}{2}\frac{0{\cdot}664}{\sqrt{\pi}}\left(\frac{l}{x_0}\right)^{\frac{1}{2}}\nu[\eta_0 g(\eta_0)]\,\xi + \ldots, \tag{4.20}$$

where
$$\eta_0 = Y = y\left(\frac{U_\infty}{\nu x_0}\right)^{\frac{1}{2}}.$$

Normalizing with a characteristic length $\delta_r = (\nu x_0/U_\infty)^{\frac{1}{2}}$ and a characteristic velocity $u_r = 0{\cdot}664U_\infty(x_0/l)^{-\frac{1}{2}}/\sqrt{\pi}$, the dimensionless stream function becomes

$$\overline{\psi} = \frac{\sqrt{\pi}}{0{\cdot}664}\left(\frac{x_0}{l}\right)^{\frac{1}{2}} Y - F(Y) + \frac{1}{2}\left(\frac{\nu}{x_0 U_\infty}\right)^{\frac{1}{2}}[Yg(Y)]\xi + O\left(\frac{\nu}{x_0 U_\infty}\right).$$

$\overline{\psi}_0$ does not appear in (2.12), (2.8)–(2.10), and the only place where the first term shows up is in $(\omega_0 + \alpha_0\overline{\psi}_0')$ of the Orr–Sommerfeld operator L. But

$$\overline{\psi}' = \frac{\sqrt{\pi}}{0{\cdot}664}\left(\frac{x_0}{l}\right)^{\frac{1}{2}} - g(Y) + O\left(\left(\frac{\nu}{x_0 U_\infty}\right)^{\frac{1}{2}}\right).$$

Parallel flow eigensolution				Non-parallel flow corrections								
				Perturbation in neutral curve							Perturbation at fixed R	
α_0	R_0	$\omega_0\times10^2$	$-c_0$	$\alpha_1\times10^3$	$\omega_1\times10^{-3}$	b_1	R_1	$c_{10}\times10^{-2}$	$c_{11}\times10^2$		$\omega_1\times10^{-2}$	$R_0+\dfrac{1}{2R_0}R_1$
0·05	2·51	0·0643	0·0128	$-3·09+2·52i$	0·687	0·832	$-5·24$	$-1·37$	$-0·0794+0·0648i$		$-0·135-0·310i$	1·467
0·075	2·31	0·197	0·0263	$-4·78+1·57i$	1·31	0·459	$-2·44$	$-1·74$	$-0·168+0·0552i$		$-0·0617-0·464i$	1·77
0·10	2·26	0·432	0·0432	$-6·13+0·390i$	2·92	0·280	$-1·43$	$-2·92$	$-0·265+0·0168i$		$0·145+0·546i$	1·94
0·125	2·27	0·781	0·0624	$-7·31-1·28i$	4·04	0·155	$-0·803$	$-3·24$	$-0·365-0·0638i$		$0·322-0·489i$	2·10
0·15	2·33	1·24	0·0829	$-8·29-3·35i$	4·45	0·0796	$-0·431$	$-2·97$	$-0·458-0·185i$		$0·411-0·366i$	2·23
0·20	2·52	2·52	0·125	$-9·56-8·32i$	4·46	0·0211	$-0·134$	$-2·23$	$-0·615-0·523i$		$0·448-0·172i$	2·49
0 25	2·80	4·22	0·169	$-9·96-14·2i$	3·94	0·0190	$-0·149$	$-1·58$	$-0·673-0·961i$		$0·413-0·237i$	2·77
0·30	3·16	6·33	0·211	$-9·59-20·9i$	3·34	0·0358	$-0·357$	$-1·11$	$-0·674-1·47i$		$0·413-0·627i$	3·10
0·375	3·86	10·1	0·270	$-7·87-31·8i$	3·28	0·0683	$-1·02$	$-0·874$	$-0·566-2·29i$		$0·638-1·81i$	3·72
0·50	5·62	17·8	0·356	$-2·77-51·0i$	5·38	0·117	$-3·70$	$-1·07$	$-0·198-3·64i$		$1·66-5·26i$	5·29
0·70	12·14	33·1	0·472	$8·60-77·5i$	$-1·50$	0·182	$-26·8$	0·215	$0·581-5·23i$		$3·75-15·0i$	11·02
0·875	43·03	49·3	0·564	$20·2-88·4i$	$-70·7$	0·251	$-466·2$	8·09	$1·30-5·69i$		$1·85-28·6i$	37·6

TABLE 4. Computational results for the two-dimensional flat-plate wake

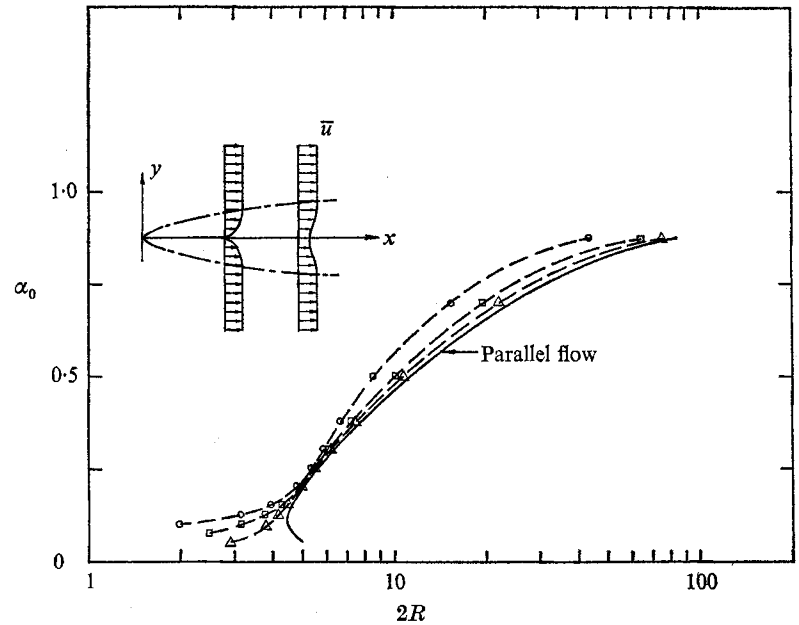

FIGURE 11. Neutral stability curves for the flat-plate wake. Non-parallel flow corrections: ○, $0{\cdot}664 \times (\pi x_0/l)^{-\frac{1}{2}} = 2$; □, 1; △, 0·5.

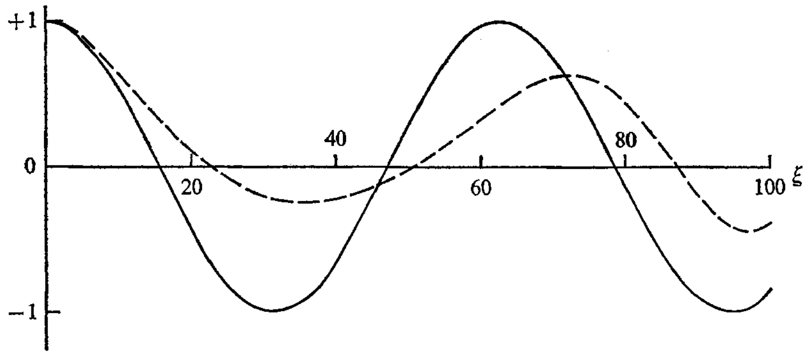

FIGURE 12. Streamwise disturbance behaviour for the flat-plate wake at $\alpha_0 = 0{\cdot}1, R_0 = 2{\cdot}26$. ——, $\mathrm{Re}\,(\exp\{i\alpha_0\xi\})$; ----, $\mathrm{Re}\,(\exp\{i\alpha\xi\})$.

Hence we replace ω_0 by $(\omega_0 + \sqrt{\pi}(x_0/l)^{\frac{1}{2}}\alpha_0/0{\cdot}664)$ in the Orr–Sommerfeld operator. Using this operator provides us with a parameter-free basic flow field, which is a decided convenience. Then in (2.5) we take

$$\bar{\psi}_0(Y) = -F(Y),$$

$$\bar{\psi}_0'(Y) = -g(Y) = -\exp\{-\tfrac{1}{4}Y^2\}, \tag{4.21}$$

$$\bar{\psi}_1(Y) = \tfrac{1}{2}Yg(Y) - \bar{\psi}_0'', \tag{4.22}$$

$$\epsilon = \left(\frac{\nu}{x_0 U_\infty}\right)^{\frac{1}{2}} = \frac{0{\cdot}664}{R(\pi(x_0/l))^{\frac{1}{2}}}, \tag{4.23}$$

where

$$R = u_r \delta_r/\nu.$$

Solutions were obtained for symmetric eigenfunctions. Table 4 gives the computational results for two-dimensional flat-plate wake. Figure 12 shows the streamwise growth of disturbance for $\alpha_0 = 0{\cdot}1$ and $R_0 = 2{\cdot}26$.

Figure 11 shows the neutral stability curve with the first-order non-parallel correction in comparison with the parallel flow theory. Curves were plotted for different values of x_0/l. The non-parallel correction is most significant at low values of α_0. The results suggest that the non-parallel flow effects render the flow unstable to long wavelength disturbances at low Reynolds numbers. The problem of the validity of the mean flow obtained from the boundary layer equation at low R should of course be remembered. At $\alpha = 0{\cdot}1$ and $R = 2{\cdot}26$, $\alpha_{1r} < 0$, and hence the disturbance wavelength will increase slightly in the downstream direction. Also, $a_{1i} > 0$, hence the neutral critical disturbance will show a slight decrease in amplitude in the downstream direction. Figure 12 shows this behaviour. Note from table 4 that somewhat different behaviour is predicted for disturbances on the upper and lower branches of the neutral stability curve.

5. Conclusion

A theory for non-parallel effects was developed formally, and applied in detail for three different laminar flows. In the case of the Blasius flow, the neutral stability curve remained almost unchanged by the non-parallel correction. For a two-dimensional laminar jet or a two-dimensional laminar flat-plate wake, the flow apparently became unstable at low Reynolds number, owing to the non-parallel effects. The non-parallel effects were not very significant at high Reynolds numbers.

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